

GEOMETRIC FLOWS AND STROMINGER SYSTEMS ¹

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Abstract

A geometric flow on $(2, 2)$ -forms is introduced which preserves the balanced condition of metrics, and whose stationary points satisfy the anomaly equation in Strominger systems. The existence of solutions for a short time is established, using Hamilton's version of the Nash-Moser implicit function theorem.

1 Introduction

The Strominger system is a system of equations for a metric on a 3-dimensional complex manifold X equipped with a nowhere vanishing holomorphic 3-form, and a Hermitian metric on a holomorphic vector bundle over X . It is of considerable interest both in physics, where it is the equation for supersymmetric compactifications of the heterotic string to a four-dimensional space-time, and in mathematics, where it is a non-Kähler generalization of a Calabi-Yau metric [34] coupled to a Hermitian-Einstein connection [5, 33]. The first mathematically rigorous solutions to the Strominger system were found by Li-Yau [21] and Fu-Yau [14, 15], and more solutions were found in [1, 2, 7, 8, 9, 10, 11, 12, 17, 23, 32]. Other solutions on physical grounds were studied by physicists in e.g. [3, 4, 6]. Generalizations of the Fu-Yau equation to higher dimensions are considered in [24, 25]. The general solution appears out of reach at the present time.

The main goal of this paper is to propose a geometric flow of $(2, 2)$ -forms, whose stationary points provide solutions of the Strominger system. We call it the Anomaly flow, as its curvature terms are the local characteristic classes arising in gravitational and Yang-Mills anomalies in string theory. In this paper, we present some evidence that the Anomaly flow may provide a viable approach to Strominger systems. In particular, we show that it preserves the balanced property of metrics, that it also admits a more conventional description as a flow of Hermitian metrics by their curvatures, and that short-time solutions always exist for small values of the string tension parameter. Estimates for solutions and criteria for long-time existence and convergence are relegated to later work.

2 The Anomaly flow

Let X be a compact 3-dimensional complex manifold, which admits a nowhere vanishing holomorphic $(3, 0)$ -form Ω . Let $E \rightarrow X$ be a holomorphic vector bundle over X . Let ω_0

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be a Hermitian metric on X , and H_0 a Hermitian metric on E . We define the Anomaly flow for the pair (X, E) to be the flow $\omega(t), H(t)$ of metrics on X and on E given by,

$$\begin{aligned}\partial_t(\|\Omega\|_\omega\omega^2) &= i\partial\bar{\partial}\omega - \frac{\alpha'}{4}(\text{Tr}(R \wedge R) - \text{Tr}(F \wedge F)) \\ H^{-1}\partial_t H &= -\Lambda F\end{aligned}\tag{2.1}$$

with initial condition $\omega(0) = \omega_0$, $H(0) = H_0$. Here α' is a fixed positive parameter, called the string tension in the physics literature. The expressions $R = (R^p_q)$ and $F = (F^\alpha_\beta)$ denote respectively the curvature of the Chern connections defined by $\omega(t)$ and $H(t)$ on X and on E . They are $(1, 1)$ -forms, valued respectively in the bundles $\text{End}(T^{1,0})$ and $\text{End}(E)$ of endomorphisms of $T^{1,0}$ and E . Our conventions are

$$[\nabla_j, \nabla_{\bar{k}}]V^p = R_{\bar{k}j}^p{}_q V^q, \quad [\nabla_j, \nabla_{\bar{k}}]\varphi^\alpha = F_{\bar{k}j}^{\alpha}{}_\beta \varphi^\beta,\tag{2.2}$$

where V^p and φ^α are respectively sections of $T^{1,0}$ and E in a local trivialization. The Hermitian form ω is defined by $\omega = i g_{\bar{k}j} dz^j \wedge d\bar{z}^k$, and a (p, q) form η has component $\eta_{\bar{k}_1 \dots \bar{k}_q j_1 \dots j_p}$ given by

$$\eta = \frac{1}{p!q!} \sum \eta_{\bar{k}_1 \dots \bar{k}_q j_1 \dots j_p} dz^{j_p} \wedge \dots \wedge dz^{j_1} \wedge d\bar{z}^{k_q} \wedge \dots \wedge d\bar{z}^{k_1}.\tag{2.3}$$

With this convention, the $(1, 1)$ -forms $R = (R^p_q)$ and $F = (F^\alpha_\beta)$ are then given by

$$R^p_q = R_{\bar{k}j}^p{}_q dz^j \wedge d\bar{z}^k, \quad F^\alpha_\beta = F_{\bar{k}j}^{\alpha}{}_\beta dz^j \wedge d\bar{z}^k.$$

We also define the pointwise inner product $\langle \cdot, \cdot \rangle_\omega$ as

$$\langle \phi, \psi \rangle_\omega = \frac{1}{p!q!} g^{\mu_1 \bar{\beta}_1} \dots g^{\mu_q \bar{\beta}_q} g^{\alpha_1 \bar{\lambda}_1} \dots g^{\alpha_p \bar{\lambda}_p} \phi_{\bar{\beta}_1 \dots \bar{\beta}_q \alpha_1 \dots \alpha_p} \overline{\psi_{\bar{\mu}_1 \dots \bar{\mu}_q \lambda_1 \dots \lambda_p}},$$

for any (p, q) forms ϕ and ψ . The Hodge operator Λ is defined as usual by $(\Lambda F)^\alpha_\beta = g^{j\bar{k}} F_{\bar{k}j}^{\alpha}{}_\beta$. Here $(g^{j\bar{k}})$ denotes the inverse $(g_{j\bar{k}})^{-1}$ of $(g_{j\bar{k}})$.

We shall also be interested in a version of the Anomaly flow for just a metric $\omega(t)$ on X . Thus let Φ_0 be a given closed smooth $(2, 2)$ -form on X . The Anomaly flow for X with given Φ_0 is defined by

$$\partial_t(\|\Omega\|_\omega\omega^2) = i\partial\bar{\partial}\omega - \frac{\alpha'}{4}(\text{Tr } R \wedge R - \Phi_0)\tag{2.4}$$

with initial condition $\omega(0) = \omega_0$.

The following simple theorem provides the motivation for the Anomaly flows:

Theorem 1 *Let X, E, Ω be as above, and consider the equations (2.1) or (2.4) on either (X, E) or X , with initial metrics ω_0 and H_0 .*

(a) The equation (2.4) is well-defined as a flow, i.e., it defines a vector field on the space of metrics, or equivalently, a vector field on the space of positive Hermitian $(2, 2)$ -forms. The flows are local, in the sense that the vector fields are given by local expressions in the underlying Hermitian metrics (or the underlying $(2, 2)$ -form).

Similarly for the equation (2.1), which defines now a vector field on the direct sum of the space of Hermitian metrics on X (or positive $(2, 2)$ -forms) with the space of Hermitian metrics on E .

(b) Assume that the Anomaly flows admit a smooth solution on some time interval $[0, T)$. If the initial metric ω_0 satisfies the balanced condition ¹

$$d(\|\Omega\|_{\omega_0}\omega_0^2) = 0 \quad (2.5)$$

then the metric $\omega(t)$ will satisfy the same balanced condition, namely $d(\|\Omega\|_{\omega_t}\omega_t^2) = 0$, for any t in its time interval $[0, T)$ of existence.

(c) Assume that the Anomaly flow for (X, E) exists for all time $[0, \infty)$, and that the initial metric satisfies the balanced-like condition (2.5). If the flow $(\omega(t), H(t))$ converges to a pair $(\omega_\infty, H_\infty)$ of Hermitian metrics on X and E , then this pair satisfies the Strominger system of equations ²

$$\begin{aligned} F_\infty \wedge \omega_\infty^2 &= 0 \\ F_\infty^{2,0} &= F_\infty^{0,2} = 0 \\ i\partial\bar{\partial}\omega_\infty &= \frac{\alpha'}{4} (\text{Tr} R_\infty \wedge R_\infty - \text{Tr} F_\infty \wedge F_\infty) \\ d(\|\Omega\|_{\omega_\infty}\omega_\infty^2) &= 0. \end{aligned} \quad (2.6)$$

Similarly, if the Anomaly flow on X with given Φ_0 converges, then the limiting metric ω_∞ satisfies

$$\begin{aligned} i\partial\bar{\partial}\omega_\infty &= \frac{\alpha'}{4} (\text{Tr}(R_\infty \wedge R_\infty) - \Phi_0) \\ d(\|\Omega\|_{\omega_\infty}\omega_\infty^2) &= 0. \end{aligned} \quad (2.7)$$

We note that the condition $d(\|\Omega\|_{\omega_\infty}\omega_\infty^2) = 0$ has been shown by Li and Yau [21] to be equivalent to the condition $d^\dagger\omega_\infty = i(\bar{\partial} - \partial)\log\|\Omega\|_{\omega_\infty}$, so that the system (2.6) is indeed equivalent to the system originally written down by Strominger [30].

We also note that the equations for H in the Strominger system are just the Hermitian-Einstein equation for a Chern unitary connection, and the flow of H in (2.1) is of course the well-known Donaldson heat flow, which is gauge equivalent to the Yang-Mills flow.

¹The usual balanced condition for a Hermitian metric ω in dimension n is $d\omega^{n-1} = 0$. For simplicity, we use the same terminology for the slight modification used in (2.5).

²It has recently been brought to our attention that the same system of equations for supersymmetric compactifications was independently proposed by C. Hull [19, 20].

Next, we consider the issue of short-time solutions for the flows, and when they would be parabolic. For a fixed metric ω , we define a modified operator $\tilde{\star}$ of the Hodge \star_ω operator as the operator from the space of $(2, 2)$ -forms $\delta\Psi$ to the space of $(1, 1)$ -forms, given by

$$\tilde{\star} \delta\Psi = \frac{1}{2\|\Omega\|_\omega} (\langle \star \delta\Psi, \omega \rangle \omega - \star \delta\Psi). \quad (2.8)$$

Next, we view the curvature tensor $R_{\bar{k}j}{}^p{}_q$ of ω as the operator Rm from the space of $(1, 1)$ -forms $\delta\omega$ into itself given by

$$Rm(\delta\omega)_{\bar{k}j} = R_{\bar{k}j}{}^{p\bar{q}}(\delta\omega)_{\bar{q}p}. \quad (2.9)$$

We can then define the following linear differential operator $\tilde{\Delta}$ of order 2 on the space of Hermitian tensors $\delta\Psi$ of type $(2, 2)$,

$$\tilde{\Delta}(\delta\Psi) = i\partial\bar{\partial}(\tilde{\star}\delta\Psi - \frac{\alpha'}{2} Rm(\tilde{\star}\delta\Psi)). \quad (2.10)$$

We note that the range of $\tilde{\Delta}$ is contained in the space of closed $(2, 2)$ -forms. We shall say that the operator $\tilde{\Delta}$ is *elliptic on the space of closed $(2, 2)$ -forms* if its symbol, restricted to the null space of the symbol of the exterior derivative d , admits only eigenvalues with strictly positive real parts. We have then the following theorem:

Theorem 2 *Let (X, Ω, E) be as before, and consider the flows (2.1) and (2.4), with initial Hermitian metrics ω_0 and H_0 on X and E respectively. If the operator $\tilde{\Delta}$ with respect to the initial metric ω_0 is elliptic on the space of closed $(2, 2)$ -forms, in the above sense, then both flows (2.1) and (2.4) admit a smooth solution on some non-trivial finite time interval.*

3 Proof of Theorem 1

Let $\Psi = \|\Omega\|_\omega \omega^2$. The essence of Theorem 1 is that ω can be recaptured from Ψ (and of course vice versa), by purely local expressions. For this we need to discuss the issue of $(n-1)$ -th root of a positive $(n-1, n-1)$ -form in some detail.

3.1 The $(n-1)$ -th root of an $(n-1, n-1)$ -form

In general, in dimension n , let Φ be a $(n-1, n-1)$ -form which is positive definite, in the sense that

$$\Phi \wedge i\eta \wedge \bar{\eta}$$

is a positive (n, n) -form for any non-zero $(1, 0)$ -form η and which equals 0 if and only if $\eta = 0$. Michelsohn [22] has shown that there exists a unique positive $(1, 1)$ -form ω with

$$\omega^{n-1} = \Phi. \quad (3.1)$$

We need a viable formula for ω , which can be obtained as follows. Let Φ be expressed as in [22, 31] by

$$\begin{aligned} \Phi = & i^{n-1}(n-1)! \sum_{k,j} (\text{sgn}(k, j)) \Phi^{k\bar{j}} dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge \widehat{dz^k} \wedge d\bar{z}^k \wedge \cdots \\ & \wedge dz^j \wedge \widehat{d\bar{z}^j} \wedge \cdots \wedge dz^n \wedge d\bar{z}^n \end{aligned} \quad (3.2)$$

where $\text{sgn}(k, j) = -1$ if $k > j$ and $\text{sgn}(k, j) = 1$ otherwise. One advantage for this representation is that $\Phi^{k\bar{j}}$ is a Hermitian matrix. Then the $(n-1)$ -th root $\omega = i g_{\bar{j}k} dz^k \wedge d\bar{z}^j$ of Φ is given by

$$g_{\bar{j}k} = (\det g) (\Phi^{-1})_{\bar{j}k}, \quad (3.3)$$

where $(\Phi^{-1})_{\bar{j}k}$ is the inverse matrix of $\Phi^{k\bar{j}}$, i.e., $\Phi^{k\bar{j}}(\Phi^{-1})_{\bar{j}\ell} = \delta^k_\ell$. To see this, we note that the entry $(\omega^{n-1})^{j\bar{k}}$ in the product ω^{n-1} is obtained by taking the product of the entries, with corresponding permutation signs and $(n-1)!$ factor, of the matrix obtained from $g_{\bar{p}q}$ by removing the j -th row and the k -th column. In other words, it is the $j\bar{k}$ cofactor of the matrix $(g_{\bar{p}q})$. The equation (3.3) is just a reformulation of this statement.

The notion of $(n-1)$ -th root is independent of any metric. Nevertheless, it can be useful to express it in terms of the Hodge star operator. If $\tilde{\omega} = i \tilde{g}_{\bar{k}j} dz^j \wedge d\bar{z}^k$ is any metric, recall that the Hodge star operator $\star_{\tilde{\omega}}$ with respect to $\tilde{\omega}$ is defined by the equation

$$\phi \wedge \overline{\Phi} = \frac{\langle \phi, (\star_{\tilde{\omega}} \Phi) \rangle_{\tilde{\omega}}}{n!} \tilde{\omega}^n \quad (3.4)$$

for any $(1, 1)$ -form $\phi = \phi_{\bar{k}j} dz^j \wedge d\bar{z}^k$. The left-hand side can be easily recognized to be

$$i^{-1} (n-1)! \phi_{\bar{k}j} \Phi^{j\bar{k}} \prod_{\ell=1}^n i dz^\ell \wedge d\bar{z}^\ell = -i \frac{\phi_{\bar{k}j} \Phi^{j\bar{k}}}{\det \tilde{g}} \frac{\tilde{\omega}^n}{n}. \quad (3.5)$$

Here we use the fact that $(\Phi^{j\bar{k}})$ is Hermitian. We can also write the right-hand side as

$$\tilde{g}^{p\bar{k}} \tilde{g}^{j\bar{q}} \phi_{\bar{k}j} \overline{(\star_{\tilde{\omega}} \Phi)_{\bar{p}q}} \frac{\tilde{\omega}^n}{n!}. \quad (3.6)$$

This implies that

$$(\star_{\tilde{\omega}} \Phi)_{\bar{p}q} = i \frac{(n-1)!}{\det \tilde{g}} \Phi^{k\bar{j}} \tilde{g}_{\bar{p}k} \tilde{g}_{j\bar{q}}. \quad (3.7)$$

As a check, we note that, since the expression in (3.5) is an (n, n) -form, the factor $\phi_{\bar{k}j} \Phi^{j\bar{k}} / \det \tilde{g}$ must be a scalar, and hence $\Phi^{k\bar{j}}$ should be interpreted as a section of $(\Lambda^{1,1})^* \otimes K_X \otimes \overline{K_X}$. This is consistent with the fact that the expression given in (3.7) is a $(1, 1)$ -form. Taking $\tilde{\omega}$ in (3.7) to be ω itself, we obtain

$$\star_{\omega} \Phi = (n-1)! \omega,$$

a formula that can also be easily seen using an orthonormal basis for ω . Henceforth, we shall suppress the subindex ω in the star operator, when the metric ω is implicit.

3.2 The relation $\Psi = \|\Omega\|_{\omega} \omega^2$

We return to the setting of a 3-fold X , equipped with a fixed nowhere vanishing $(3, 0)$ -form Ω . We will use the notation $|\Omega|^2 = \Omega \overline{\Omega}$ and $\|\Omega\|_{\omega}^2 = \Omega \overline{\Omega} (\det g)^{-1}$. If Ψ is any positive $(2, 2)$ -form, we claim that there is a unique positive $(1, 1)$ -form ω so that $\Psi = \|\Omega\|_{\omega} \omega^2$.

Indeed, this equation determines the norm $\|\Omega\|_{\omega}$, since taking determinants gives

$$\det \Psi = (|\Omega|^2 (\det g)^{-1})^{3/2} (\det g)^2 \quad (3.8)$$

and hence

$$(\det g)^{\frac{1}{2}} = \frac{\det \Psi}{|\Omega|^3}. \quad (3.9)$$

This determines $\det g$ in terms of Ψ , and hence $\|\Omega\|_{\omega}$ in terms of Ψ . We can then obtain ω as the square root of the positive $(2, 2)$ -form $\|\Omega\|_{\omega}^{-1} \Psi$. The relations (3.3) and (3.7) become

$$g_{\bar{j}k} = \frac{\det \Psi}{|\Omega|^2} \Psi_{\bar{j}k}^{-1}, \quad \star_{\omega} \Psi = 2 \|\Omega\|_{\omega} \omega. \quad (3.10)$$

3.3 Proof of Theorem 1, Part (a)

It is now straightforward to relate the variations of ω to the variations of Ψ . Differentiating the first equation on the left side of (3.10) gives

$$\delta g_{\bar{j}k} = \frac{1}{\|\Omega\|_{\omega} \det g} (g_{\bar{q}p} \delta \Psi^{p\bar{q}} g_{\bar{j}k} - g_{\bar{j}p} \delta \Psi^{p\bar{q}} g_{\bar{q}k}). \quad (3.11)$$

In intrinsic notation, using (3.7), this can be rewritten as

$$\delta \omega = \frac{1}{2 \|\Omega\|_{\omega}} (\langle \star \delta \Psi, \omega \rangle \omega - \star \delta \Psi) = \tilde{\star} \delta \Psi, \quad (3.12)$$

in view of the definition of the operator $\tilde{\star}$.

In particular, along the Anomaly flow, we can replace $\delta\omega$ by $\partial_t\omega$ and $\delta\Psi$ by $\partial_t\Psi = \partial_t(\|\Omega\|_\omega\omega^2)$. We obtain in this way an equation giving $\partial_t\omega$ in terms of ω and its curvature, which is the more conventional description of a geometric flow of Hermitian metrics. Equivalently, the flows can be written as flows of $(2, 2)$ -forms Ψ , given by a local vector field on the space of positive Hermitian $(2, 2)$ -forms.

3.4 Proof of Theorem 1, Parts (b) and (c)

The only non-trivial part of Theorem 1 is the conceptual part due to the issue of taking square roots. Once this issue has been clarified, the proof is straightforward.

Part (b) follows immediately from the fact that the right hand sides of (2.1) and (2.4) are always closed forms. This follows itself from the fact that $d\partial\bar{\partial}\omega = d^2\bar{\partial}\omega = 0$, and that both $\text{Tr}(R \wedge R)$ and $\text{Tr}(F \wedge F)$ are well-known closed representatives of the Chern classes $c_2(T^{1,0})$ and $c_2(E)$ of the bundles $T^{1,0}(X)$ and E . Thus

$$\partial_t(d(\|\Omega\|_\omega\omega^2)) = d\partial_t(\|\Omega\|_\omega\omega^2) = 0, \quad (3.13)$$

and $d(\|\Omega\|_\omega\omega^2) = 0$ for all time t , if $d(\|\Omega\|_\omega\omega^2) = 0$ at $t = 0$.

Part (c) follows immediately from Part (b), which guarantees that the balanced-like equation in the Strominger systems is satisfied for all time. The equation $F_\infty^{2,0} = F_\infty^{0,2} = 0$ is automatic for all Chern connections. The other equations are obvious consequences of the fact that the limiting metric ω_∞ must be stationary.

4 Short-time existence and parabolicity

To obtain short time existence, we consider the Anomaly flows as an evolution equation. Let V be a smooth vector bundle over a compact manifold X , and consider the equation,

$$\partial_t\psi = \mathcal{E}(\psi) \quad (4.1)$$

where $\mathcal{E}(\psi)$ is a non-linear differential operator of order 2, acting on the sections ψ of V . If the eigenvalues of the symbol $\sigma(\delta\mathcal{E}(\psi))(x, \xi)$ of the linearization $\delta\mathcal{E}$ of \mathcal{E} have strictly positive real parts for $\xi \neq 0$, $(x, \xi) \in T_*(X)$, then the equation is parabolic, and the evolution equation with initial data ψ admits a solution for short-time. More generally, we have the following version of the Nash-Moser theorem, as formulated by Hamilton [18], and applied by him to show the existence of short-time solution for the Ricci flow:

Lemma 1 *Let $L : C^\infty(V) \rightarrow C^\infty(W)$ be a linear differential operator of order 1 with values in another vector bundle W . Assume that*

- (a) *The composition $Q(\Psi) = L(\Psi)\mathcal{E}(\Psi)$ is a differential operator of order at most 1;*
- (b) *The symbol $\sigma(\delta\mathcal{E}(\Psi))(x, \xi)$ has eigenvalues with strictly positive real parts when restricted to the kernel of the symbol $\sigma(\delta L(\Psi))(x, \xi)$.*

Then the initial value problem (4.1) admits a unique solution for short time.

4.1 Proof of Theorem 2

We consider first the notationally simpler case of the Anomaly flow (2.4) on X . In this case, the bundle V is the bundle $\Lambda^{2,2}(X)$ of $(2,2)$ -forms, the sections ψ are the $(2,2)$ -forms Ψ , and $\mathcal{E}(\psi)$ is given by the right hand side of (2.4). The linearization of $i\partial\bar{\partial}\omega$ follows readily from the equation (3.12),

$$(i\partial\bar{\partial}\delta\omega) = i\partial\bar{\partial}\left(\frac{1}{2\|\Omega\|_\omega}(\langle\star\delta\Psi, \omega\rangle\omega - \star\delta\Psi)\right) = i\partial\bar{\partial}(\tilde{\star}\delta\Psi), \quad (4.2)$$

in the notation of (2.8). Next, we determine the linearization of the curvature terms in the Anomaly flow. The variation of the curvature F of a unitary Chern connection under a variation δH of the Hermitian metric is given by (see e.g. [29]),

$$\delta F = \bar{\partial}\partial^H(H^{-1}\delta H), \quad (4.3)$$

where ∂^H denotes the covariant exterior derivative in the unbarred directions. In particular,

$$\delta\text{Tr}(F \wedge F) = 2\text{Tr}(F \wedge \bar{\partial}\partial^H(H^{-1}\delta H)). \quad (4.4)$$

In view of the Bianchi identity, $d^H F = 0$, this can be rewritten as

$$\delta\text{Tr}(F \wedge F) = -2\partial\bar{\partial}\text{Tr}(FH^{-1}\delta H). \quad (4.5)$$

Specializing to the case where the vector bundle is $T^{1,0}(X)$, we obtain

$$\delta\text{Tr}(R \wedge R) = -2\partial\bar{\partial}\left(R_{\bar{k}j}^{\alpha\bar{\beta}}\delta g_{\bar{\beta}\alpha}dz^j \wedge d\bar{z}^k\right).$$

It follows that

$$\begin{aligned} \delta\text{Tr}(R \wedge R) &= 2i\partial\bar{\partial}\left(R_{\bar{k}j}^{\alpha\bar{\beta}}(\delta\omega)_{\bar{\beta}\alpha}dz^j \wedge d\bar{z}^k\right) \\ &= 2i\partial\bar{\partial}(Rm(\delta\omega)) = 2i\partial\bar{\partial}(Rm(\tilde{\star}\delta\Psi)) \end{aligned} \quad (4.6)$$

where we view the Riemann curvature tensor as an operator on $(1,1)$ -forms, as explained in §2. Combining the formulas (4.2) and (4.6) gives the linearization of $\mathcal{E}(\Psi)$ in the case of the Anomaly flow on X ,

$$\delta\mathcal{E}(\delta\Psi) = \tilde{\Delta}(\delta\Psi). \quad (4.7)$$

We apply Hamilton's version of the Nash-Moser implicit function theorem with the choice $L = d$ on $(2,2)$ -forms. Because the right hand side \mathcal{E} of the Anomaly flow is a closed $(2,2)$ -form, we do have $L\mathcal{E} = 0$. The condition of ellipticity of the operator $\tilde{\Delta}$ restricted to the space of closed $(2,2)$ -forms, as formulated in §2, is precisely the condition which allows Hamilton's version of the Nash-Moser implicit function theorem to apply. Thus the existence of solutions to the equation (2.4) for short-time follows.

The case of the Anomaly flow on (X, E) can be treated in the same manner. We view the flow as of the form

$$\partial_t \Psi = \mathcal{E}(\Psi, H), \quad \partial_t H = \mathcal{F}(\Psi, H) \quad (4.8)$$

with the pair (Ψ, H) given by sections of the direct sum of the bundle of $(2, 2)$ -forms with the bundle of Hermitian quadratic forms on E . Clearly \mathcal{E} and \mathcal{F} are non-linear differential operators of order 2. Applying the formulas for variations of curvature to the bundles $T^{1,0}$ and E , we find

$$\begin{aligned} \delta \mathcal{E} &= \partial \bar{\partial} \left(i \delta \omega + \frac{\alpha'}{2} \left(\text{Tr}(Rg^{-1} \delta g) - \text{Tr}(FH^{-1} \delta H) \right) \right) \\ \delta \mathcal{F} &= -H \wedge \bar{\partial} \partial^H (H^{-1} \delta H) - H \delta g^{j\bar{k}} F_{\bar{k}j} - \delta H \wedge F, \end{aligned} \quad (4.9)$$

where $F_{\bar{k}j}$ is viewed as a $(1, 1)$ -form with valued in the bundle of endomorphisms of E . It follows that the symbols of the linearization of \mathcal{E} and \mathcal{F} are given by

$$\begin{aligned} \sigma(\delta \mathcal{E}) : (\delta \Psi, \delta H) &\rightarrow \sigma(\delta \mathcal{E}_X)(\delta \Psi) - \frac{\alpha'}{2} \xi \wedge \bar{\xi} \wedge (\text{Tr}(FH^{-1} \delta H)) \\ \sigma(\delta \mathcal{F}) : (\delta \Psi, \delta H) &\rightarrow |\xi|^2 \delta H \end{aligned} \quad (4.10)$$

where we have temporarily denoted by \mathcal{E}_X the expression on the right hand side of the Anomaly flow on X .

We choose the operator L of the Nash-Moser implicit function theorem as

$$L(\Psi, H) = (d\Psi, 0).$$

The right hand side $\mathcal{E}(\Psi, H)$ is again always a closed form, so we do have $L(\mathcal{E}, \mathcal{F}) = 0$. Furthermore, the above formulas show that the symbol of the combined system $(\mathcal{E}, \mathcal{F})$ is a triangular block matrix, with the blocks on the diagonal given by $\sigma(\delta \mathcal{E}_X)$ acting on $\delta \Psi$ and $\sigma(\delta \mathcal{F})$ acting on δH . The first block has already been shown to have eigenvalues with positive real parts when restricted to the kernel of d , while the second is manifestly strictly positive. So the short-time existence of solutions to the Anomaly flow on (X, E) follows, and the proof of Theorem 2 is complete.

4.2 Discussion of the parabolicity condition

The linearization operator $\tilde{\Delta}$ and its ellipticity restricted to the space of $(2, 2)$ closed forms is of considerable importance, since $\tilde{\Delta}$ controls the evolution of derivative quantities of Ψ and ω such as the curvature.

First we observe that the ellipticity condition can also be reformulated in terms of operators on $(1, 1)$ -forms. It suffices to set $\delta \Psi = \star \delta T$, for $(1, 1)$ -forms T . Then the

ellipticity of $\tilde{\Delta}$ on the space of closed $(2, 2)$ -forms is equivalent to the ellipticity of the operator

$$\star \delta T \rightarrow \tilde{\Delta}(\star \delta T) \quad (4.11)$$

restricted to $(1, 1)$ -forms δT satisfying $d^\dagger \delta T = 0$, defined similarly in terms of the eigenvalues of its symbol, restricted to the kernel of the symbol of d^\dagger . This follows at once from the well-known formula $d^\dagger = -\star d \star$.

Next, it is instructive to work out the ellipticity condition of the operator $\tilde{\Delta}$ restricted to the space of $(2, 2)$ -forms more explicitly. The symbol $\sigma_{\tilde{\Delta}}$ of $\tilde{\Delta}$ is given by

$$\sigma_{\tilde{\Delta}}(\xi) : \delta \Psi \rightarrow i\xi \wedge \bar{\xi} \wedge (\tilde{\star} \delta \Psi - \frac{\alpha'}{2} Rm(\tilde{\star} \delta \Psi)) \quad (4.12)$$

where $\xi \in T^*(X)$. The following lemma is useful:

Lemma 2 *For all $\delta \Psi$ in the kernel of the symbol of the exterior derivative d on $(2, 2)$ -forms, we have*

$$i\xi \wedge \bar{\xi} \wedge \tilde{\star} \delta \Psi = \frac{1}{2\|\Omega\|_\omega} |\xi|^2 \delta \Psi. \quad (4.13)$$

Proof. Without loss of generality, we can choose coordinates so that $g_{j\bar{k}} = \delta_{jk}$, and by an additional orthogonal rotation if necessary, that $\xi = (\xi_1, 0, 0)$. Let Φ denote the left-hand side of (4.13). Using the coordinate expression (3.11) and the convention (3.2) for components of a $(2, 2)$ -form, we find

$$\begin{aligned} \Phi^{k\bar{1}} &= 0, & k &= 1, 2, 3 \\ \Phi^{k\bar{k}} &= \frac{1}{2\|\Omega\|_\omega} |\xi|^2 (\delta \Psi^{1\bar{1}} + \delta \Psi^{k\bar{k}}), & k &= 2, 3 \\ \Phi^{2\bar{3}} &= \frac{1}{2\|\Omega\|_\omega} |\xi|^2 \delta \Psi^{2\bar{3}}. \end{aligned} \quad (4.14)$$

Now the kernel of the exterior derivative d on $(2, 2)$ -forms is given by forms $\delta \Psi$ satisfying

$$\xi_j \delta \Psi^{j\bar{k}} = 0 \quad k = 1, 2, 3. \quad (4.15)$$

In the given coordinate system, this reduces to $\delta \Psi^{1\bar{k}} = 0$ for $k = 1, 2, 3$. Using these relations, we can rewrite the above identities as $\Phi = \frac{1}{2}\|\Omega\|_\omega^{-1} |\xi|^2 \delta \Psi$. The lemma is proved.

This allows us to identify immediately an important and quite general situation where the ellipticity condition, and hence the existence of short-time solutions, holds:

Proposition 1 *Consider the operator*

$$\delta \Psi \rightarrow -i\xi \wedge \bar{\xi} \wedge \frac{\alpha'}{2} Rm(\langle \star \delta \Psi, \omega \rangle \omega - \star \delta \Psi) \quad (4.16)$$

restricted to the kernel of the symbol of the exterior derivative d on $(2, 2)$ -forms. If it has operator norm $< |\xi|^2$ for any $\xi \neq 0$, then the Anomaly flows with ω as initial metric admit smooth solutions for at least a short time.

5 Remarks

We conclude with a few remarks.

(1) The balanced condition

One of the major challenges in Strominger systems is that, even with the metric H on the vector bundle E fixed, it is a system in the metric ω , in the sense that both the anomaly equation and the balanced-like condition have to be satisfied. One natural approach is to try and solve the anomaly equation with a particular ansatz which guarantees that the metric is balanced. One possible ansatz is the very general one proposed by Fu-Wang-Wu [13], Tosatti-Weinkove [31] and Popovici [28]

$$\omega^2 = \omega_0^2 + i\partial\bar{\partial}(u\tilde{\omega}) \quad (5.1)$$

Here ω_0 is a balanced metric, and $\tilde{\omega}$ is an arbitrary $(1, 1)$ -form, and the form ω^2 is required to be positive. Tosatti and Weinkove have also shown in [31] how to find a single balanced metric ω_0 .

Another ansatz is the one by Fu and Yau [14], in the special case of Goldstein-Prokushkin manifolds discussed below (see eq. (5.2) below). But unlike in the Kähler case, where metrics can be represented by a potential, there does not appear to be a uniquely compelling ansatz for balanced metrics at the present time. Thus, a very attractive feature of the Anomaly flows is that they guarantee that the metrics be balanced, without appealing to any particular ansatz.

(2) Toric fibrations over $K3$ surfaces

The first non-perturbative solution of a Strominger system was found by Fu-Yau [14], as a toric fibration over a $K3$ surface. More specifically, Goldstein and Prokushkin [16] had shown how to construct a toric fibration $\pi : X \rightarrow S$, given a Calabi-Yau surface (S, ω_S) with Ricci-flat Kähler metric $\omega_S = i(g_S)_{\bar{k}j}dz^j \wedge d\bar{z}^k$, and two anti-self-dual $(1, 1)$ -forms $\kappa_1, \kappa_2 \in 2\pi H^2(S, \mathbf{Z})$. Furthermore, there is a $(1, 0)$ -form θ on X so that $\partial\theta = 0$, $\bar{\partial}\theta = \pi^*(\kappa_1 + i\kappa_2)$, and if Ω_S is a non-vanishing holomorphic $(2, 0)$ -form on S , then the non-vanishing holomorphic $(3, 0)$ -form on X is given by $\Omega = \theta \wedge \pi^*(\Omega_S)$. The $(1, 1)$ -form $\omega_0 = \pi^*(\omega_S) + i\theta \wedge \bar{\theta}$ is a balanced metric, $d\omega_0^2 = 0$ and also satisfies $\|\Omega\|_{\omega_0} = 1$. This implies that $d(\|\Omega\|_{\omega_0}\omega_0^2) = 0$.

Under suitable cohomological conditions on the class κ_1 and κ_2 , Fu and Yau found a solution of the Strominger system under the following ansatz,

$$\omega_u = \pi^*(e^u\omega_S) + i\theta \wedge \bar{\theta} \quad (5.2)$$

where u is a scalar function on S . Metrics of the form ω_u are automatically balanced and satisfy $\|\Omega\|_{\omega_u} = e^{-u}$. Here we observe that another hint that the Anomaly flow is a

natural flow, is that it preserves the Fu-Yau ansatz (5.2). Indeed, under the cohomological conditions mentioned previously, Fu and Yau have shown that

$$i\partial\bar{\partial}\omega_u - \frac{\alpha'}{4}(\text{Tr}(R_u \wedge R_u) - \text{Tr}(F \wedge F)) = i\partial\bar{\partial}(e^u\omega_S - \alpha'e^{-u}\rho) - \frac{\alpha'}{2}\partial\bar{\partial}u \wedge \partial\bar{\partial}u + \tilde{\mu}\frac{\omega_S^2}{2!}$$

for some smooth function $\tilde{\mu} : S \rightarrow \mathbf{R}$ and smooth real $(1, 1)$ form ρ defined on S and given by

$$\rho = -\frac{i}{2}\text{Tr}(\bar{\partial}B \wedge \partial B^* g_S^{-1}), \quad (5.3)$$

where B is defined on S and only depends on (S, ω_S) and κ_1, κ_2 . Thus $\partial_t(\|\Omega\|_{\omega_u}\omega_u^2)$ is the pull-back of a $(2, 2)$ -form on S . Since

$$\|\Omega\|_{\omega_u}\omega_u^2 = \omega_0^2 + (e^u - 1)\omega_S^2 \quad (5.4)$$

we see that an evolution of $\|\Omega\|_{\omega_u}\omega_u^2$ by a term proportional to ω_S^2 is just an evolution of the conformal factor u , and ω_u still satisfies the same ansatz. In fact, the Anomaly flow is immediately seen to be equivalent to the equation

$$\partial_t u = \frac{e^{-u}}{2} \left(\Delta e^u + \alpha'\sigma_2(i\partial\bar{\partial}u) - 2\alpha'\frac{i\partial\bar{\partial}(e^{-u}\rho)}{\omega_S^2} + \tilde{\mu} \right), \quad (5.5)$$

where the Laplacian and $\sigma_2(i\partial\bar{\partial}u)\omega_S^2 = i\partial\bar{\partial}u \wedge i\partial\bar{\partial}u$ are with respect to the metric ω_S . This equation may be of interest in its own right. Its parabolicity is equivalent to the ellipticity of the right hand side, which is the ellipticity condition imposed by Fu and Yau ([14], eqs. (7.3) and (8.1)). Explicitly, denote

$$\begin{aligned} E(u) &= i\partial\bar{\partial}\omega_u - \frac{\alpha'}{4}(\text{Tr}(R_u \wedge R_u) - \text{Tr}(F \wedge F)) \\ &= i\partial\bar{\partial}(e^u\omega_S - \alpha'e^{-u}\rho) + \frac{\alpha'}{2}i\partial\bar{\partial}u \wedge i\partial\bar{\partial}u + \tilde{\mu}\frac{\omega_S^2}{2!}. \end{aligned}$$

Its symbol is given by

$$\sigma(\delta E) : \delta u \rightarrow i\xi \wedge \bar{\xi} \wedge (\delta u) (e^u\omega_S + \alpha'e^{-u}\rho + \alpha'i\partial\bar{\partial}u). \quad (5.6)$$

The ellipticity condition reduces to the following condition on u

$$e^u\omega_S + \alpha'e^{-u}\rho + \alpha'i\partial\bar{\partial}u > 0. \quad (5.7)$$

We now compare this condition with Proposition 1. The condition

$$|\alpha'Rm| \ll 1, \quad (5.8)$$

implies short-time existence for the Anomaly flow by Proposition 1. In [14], Fu and Yau computed the curvature of the metric ω_u (5.2). Fixing a point $p \in X$, they constructed a frame of holomorphic vector fields such that at p ,

$$g_u = \begin{pmatrix} e^u g_S & 0 \\ 0 & 1 \end{pmatrix} \quad Rm = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \quad (5.9)$$

where the entries R_{jk} are given by

$$R_{11} = R_S - \partial\bar{\partial}u I + e^{-u} \bar{\partial}B \wedge \partial B^* g_S^{-1} \quad (5.10)$$

$$R_{12} = -\nabla \bar{\partial}B + \partial u \wedge \bar{\partial}B \quad (5.11)$$

$$R_{21} = \bar{\partial}(e^{-u} \partial B^* g_S^{-1}) \quad (5.12)$$

$$R_{22} = e^{-u} (\partial B^* g_S^{-1}) \wedge \bar{\partial}B. \quad (5.13)$$

Here $B = (\phi_1, \phi_2)^T$ is a column vector of locally defined functions ϕ_i on S , and $\bar{\partial}B$ is globally defined on S . Using the definition (5.3) of ρ and the fact that $\text{Tr} R_S = 0$, we see that $|\alpha' Rm|_{\omega_u} \ll 1$ implies

$$|2\alpha' e^{-2u} \rho|_{\omega_S} \ll 1, \quad |2\alpha' e^{-u} \partial\bar{\partial}u|_{\omega_S} \ll 1. \quad (5.14)$$

It is clear that (5.14) implies (5.7).

Update:

We note that there has been very recently significant progress on the Anomaly flows in several directions. In particular, an explicit formula for $\partial_t g_{\bar{k}j}$ has now been obtained [26] which overcomes the difficulty of the flow being originally formulated as a flow of $(2,2)$ -forms. Furthermore, the convergence of the Anomaly flow on Goldstein-Prokushkin fibrations has been established in [27], for both $\alpha' > 0$ and $\alpha' < 0$, thus unifying the results of Fu and Yau [14, 15]. These constitute strong evidence that the Anomaly flows should provide an efficient approach to Strominger systems.

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